BOUNDED SUBGROUPS OF RELATIVELY FINITELY PRESENTED GROUPS

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Abstract. In this paper we prove that every infinite bounded subgroup of a relatively finitely presented group is already conjugated to a subgroup of a parabolic group if the relative Dehn function is well defined.

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1. Introduction

Given a finitely generated group G which has a finite relative presentation

$$\langle X,\ \mathcal{H} \mid S = 1 \text{ for } S \in S_\lambda,\ R = 1 \text{ for } R \in \mathcal{R}\rangle \tag{1.1}$$

relative to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$, we want to determine the subgroups of G which are bounded with respect to the relative metric $d_{X \cup \mathcal{H}}$.

It turns out that under the mild assumption that the relative Dehn function is well defined, there are only trivial examples of such subgroups.

Theorem *Let G be a finitely generated group with a finite generating set X. If G is finitely presented with respect to a set of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ such that the relative Dehn function is well defined. Let $K \leqslant G$ be a subgroup which is bounded with respect to the relative metric $d_{X \cup \mathcal{H}}$. Then K is either finite or conjugated to an infinite subgroup of a parabolic group.*

2. Relatively finitely presented groups

In this section we provide the necessary notation for the study of relatively finitely presented groups which was introduced by Osin [Osi06].

Definition 2.1. [Osi06, 2.1] Let G be a group and let $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ be a collection of subgroups of G. A subset $X \subset G$ is called a *relative generating set* of G with respect to H_Λ if G is generated by $\bigcup_{\lambda \in \Lambda} H_\lambda \cup X$.

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For every index $\lambda \in \Lambda$ we fix an isomorphic copy $\widetilde{H}_\lambda$ of H_λ such that the union $\mathcal{H} = \bigcup_{\lambda \in \Lambda} (\widetilde{H}_\lambda \backslash \{1\})$ is disjoint. Further we will always assume that the relative generating set satisfies $X = X^{-1}$ and $X \cap \widetilde{H}_\lambda = \varnothing$ for every $\lambda \in \Lambda$. Then G is a quotient of the group

$$F = (*_{\lambda \in \Lambda} \widetilde{H}_\lambda) * F(X).$$

Definition 2.2. [Osi06, 2.2] In the above situation let ε be the canonical projection from F to G. Let $\mathcal{R} \subset (X \cup \mathcal{H})^*$ be a subset such that the normal closure $\langle\langle \mathcal{R} \rangle\rangle^F$ of $\mathcal{R}$ in F satisfies $\langle\langle \mathcal{R} \rangle\rangle^F = ker(\varepsilon)$. Then we say that G has the *relative presentation*

$$\langle X,\ \mathcal{H} \mid S = 1 \text{ for } S \in S_\lambda,\ R = 1 \text{ for } R \in \mathcal{R} \rangle \tag{2.1}$$

with respect to H_Λ where S_λ is the set of all words over the alphabet $\widetilde{H}_\lambda \backslash \{1\}$ which represent the identity in H_λ. The subgroups H_λ are called *parabolic*.

In the following we will always assume that the set $\mathcal{R}$ is *symmetrized*. That is, $\mathcal{R}$ is closed under taking cyclic permutations and inverses.

Definition 2.3. [Osi06, 2.3] The relative presentation (2.1) of G is called finite if the sets $\mathcal{R}$ and X are finite. If G has a finite presentation relative to a collection of subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ we say that G is *finitely presented relative to* H_Λ.

Let w be a word in $(X \cup \mathcal{H})^*$. In the following we denote the word length of w by $\| w \|$ and the image of w in G by $\overline{w}$. If Y is an ordinary generating set of G we write $|w|_Y$ for the length of $\overline{w} \in G$ with respect to the word metric d_Y.

Definition 2.4. [Osi06, 1.2] A function $f : \mathbb{N} \to \mathbb{N}$ is a *relative isoperimetric function* of the relative presentation (2.1) if for every word $w \in (X \cup \mathcal{H})^*$ of length at most n which represents the identity 1 in G, there is an expression of the form

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i f_i, \tag{2.2}$$

with $f_i \in F$, $R_i \in \mathcal{R}$ and $k \leqslant f(n)$. The minimal relative isoperimetric function of the relative presentation (2.1) is called the relative Dehn function of G with respect to H_Λ and will be denoted by $\delta^{rel}_{G,H_\Lambda}$. We note that the relative Dehn function does not have to exist since the set $(X \cup \mathcal{H})^*$ can contain infinitely many reduced words of a given length which represent the identity in G. If the relative Dehn function exists we will say that $\delta^{rel}_{G,H_\Lambda}$ is well defined.

Remark 2.5. We want to emphasize that the sets X and $\mathcal{H}$ are disjoint, but can represent the same elements in G. This may lead to the situation that there are several edges different two vertices in $\Gamma(G, X \cup \mathcal{H})$.

Definition 2.6. [Osi06, 2.18] Let $w \in (X \cup \mathcal{H})^*$ be a word. A subword v of w is an H_λ-subword if it consists of letters of $\widetilde{H}_\lambda$. An H_λ-subword v is called H_λ-syllable if it is not contained properly in another H_λ-subword. By a cyclic word w we mean the set of all cyclic permutations of w. A subword v of a cyclic word w is an H_λ-subword, if it is an H_λ-subword of a cyclic permutation of w. Accordingly, a maximal H_λ-subword of a cyclic word w is an H_λ-syllable.

Definition 2.7. [Osi06, 2.19] Let q be a path in $\Gamma(G, X \cup \mathcal{H})$. A subpath p of q is an H_λ-subpath, if the label of p is an H_λ-subword of the label of q. Here we always assume that the subpath p is endowed with the orientation induced from q. Let

$\mu(q)$ denote the label of q. An H_λ-component of q is an H_λ-subpath p such that $\mu(p)$ is an H_λ-syllable of $\mu(q)$. If q is a cyclic path in $\Gamma(G, X \cup \mathcal{H})$, then a subpath p of q is an H_λ-subpath, if $\mu(p)$ is a subword of the cyclic word $\mu(q)$. In this case an H_λ-component of q is an H_λ-subpath p of q such that $\mu(p)$ is an H_λ-syllable of the cyclic word $\mu(q)$.

Definition 2.8. [Osi06, 2.20] Two H_λ-components p_1 and p_2 of a path q (cyclic or not) in $\Gamma(G, X \cup \mathcal{H})$ are said to be connected, if there is a path c in $\Gamma(G, X \cup \mathcal{H})$ which connects a vertex of p_1 with a vertex of p_2 and $\mu(c)$ consists of letters of $\widetilde{H}_\lambda$. Two H_λ-syllables u, v of a word $w \in (X \cup \mathcal{H})^*$ are connected, if the corresponding components of a path with the label w are connected in $\Gamma(G, X \cup \mathcal{H})$.

Definition 2.9. [Osi06, 2.21] An H_λ-component p of a path q (cyclic of not) is isolated, if there are no further H_λ-components of q which are connected to p. An isolated H_λ-syllable of a word $w \in (X \cup \mathcal{H})^*$ is defined analogously.

Definition 2.10. [Osi06, 2.24] A relative presentation (2.1) of a group G is reduced, if every relation $R \in \mathcal{R}$ has the shortest length among all words in $(X \cup \mathcal{H})^*$ which represent the same element in

$$F = (*_{\lambda\in\Lambda} \widetilde{H}_\lambda) * F(X).$$

In particular every H_λ-syllable of every element $R \in \mathcal{R}$ consists of a single letter of $\widetilde{H}_\lambda$.

Definition 2.11. [Osi06, 2.25] For a relative presentation (2.1) and $\lambda \in \Lambda$ let Ω_λ be the set of all elements $h \in H_\lambda$, for which there is a relation $R \in \mathcal{R}$ and an H_λ-syllable v of R such that v represents the element h. Let further be $\Omega = \bigcup_{\lambda\in\Lambda} \Omega_\lambda$. We note that Ω is finite if $\mathcal{R}$ is finite.

The following lemma allows us to switch between the metrics d_X and $d_{X\cup\mathcal{H}}$.

Lemma 2.12. [Osi06, 2.27] *Let G be a relatively finitely presented group with respect to a collection of subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$. Let*

$$\langle X,\ \mathcal{H} \mid S = 1 \text{ for } S \in S_\lambda,\ R = 1 \text{ for } R \in \mathcal{R}\rangle \tag{2.3}$$

be a reduced, relative presentation of G. Let further q be a cyclic path of length n in $\Gamma(G, X \cup \mathcal{H})$ and $p_1, \ldots, p_k$ a set of isolated H_λ-components of q. Then

$$\overline{\mu(p_i)} \in \langle\Omega_\lambda\rangle \text{ for every } i \in \{1, \ldots, k\}. \tag{2.4}$$

Further $\sum_{i=1}^{k} |\overline{\mu(p_i)}|_{\Omega_\lambda} \leqslant M\delta^{rel}_{G,H_\Lambda}(n)$, where $M = \max_{R\in\mathcal{R}} ||R||$.

In the following we will identify a word $w \in (X \cup \mathcal{H})^*$ with the path γ in $\Gamma(G, X \cup \mathcal{H})$ which starts at the identity and has the label $\mu(\gamma) = w$. The next theorem will often be used implicitly.

Theorem 2.13. [Osi06, 2.34] *Let*

$$\langle X_1,\ \mathcal{H} \mid S = 1 \text{ for } S \in S_\lambda,\ R = 1 \text{ for } R \in \mathcal{R}_1\rangle \tag{2.5}$$

and

$$\langle X_2,\ \mathcal{H} \mid S = 1 \text{ for } S \in S_\lambda,\ R = 1 \text{ for } R \in \mathcal{R}_2\rangle \tag{2.6}$$

be two finite relative presentations of a group G with respect to a collection of subgroups $H_\lambda = \{H_\lambda \mid \lambda \in \Lambda\}$. Let further δ_1 and δ_2 be the corresponding relative

Dehn functions. If δ_1 is well defined, then δ_2 is also well defined and there are constants C, K, L satisfying

$$\delta_1(n) \leqslant C\delta_2(Kn) + Ln \text{ and } \delta_2(n) \leqslant C\delta_1(Kn) + Ln \text{ for every } n \in \mathbb{N}.$$

3. Dichotomy of infinite subgroups

To simplify the following arguments we will introduce regular neighbourhoods of paths in Cayley graphs. This definition is motivated by the corresponding definition of regular neighbourhoods in topology.

Definition 3.1. Let G be a group and let X be a generating set of G. A word $w = x_{i_1} \dots x_{i_n} \in X^*$ has a regular neighbourhood in $\Gamma(G, X)$, if for some (equivalently any) path γ in $\Gamma(G, X)$ with the label w, two vertices in γ can be connected by an edge if and only if they are already consecutive vertices in γ. This means that every subword v of w of length at least 2 satisfies $|v|_X \geqslant 2$ when considered as an element of G.

Definition 3.2. Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. A sequence of words $(w_n)_{n\in\mathbb{N}}$ over $X \cup \mathcal{H}$ satisfies the alternating growth condition, in the following ag-condition, if the words $w_n = w_1^{(n)} \dots w_{k_n}^{(n)}$ satisfy the following conditions:

1) $k_n \geqslant 2$ for every $n \in \mathbb{N}$.
2) Every word w_n has a regular neighbourhood in $\Gamma(G, X \cup \mathcal{H})$.
3) Two consecutive letters $w_i^{(n)}$ and $w_{i+1}^{(n)}$ are never lying in X, or in the same group $\widetilde{H}_\lambda$.
4) The initial letter $w_1^{(n)}$ does not lie in X.
5) Every element $w_i^{(n)} \in \mathcal{H}$ satisfies $|w_i^{(n)}|_X \geqslant n$ for every $n \in \mathbb{N}$.
6) The elements $w_i^{(n)} \in \widetilde{H}_\lambda$ do not represent elements of the set
$$X \cup \bigcup_{\mu\in\Lambda\setminus\{\lambda\}} H_\mu \subset G.$$

Remark 3.3. If we replace the 5. condition with the condition that the sequence $(|w_i^{(n)}|_X)_{n\in\mathbb{N}}$ is unbounded, then, by restriction to a subsequence, it can always be made sure that the 5. and 6. condition is satisfied. To see this, we note that on the one hand the index set Λ is finite since G is finitely generated ([Osi06, Corollary 2.48]) and on the other hand the intersection $H_\lambda \cap H_\mu$ is finite for $\lambda \neq \mu$ since the relative Dehn function was supposed to be well defined ([Osi06, Proposition 2.36]).

In the following this remark is going to be used frequently without mentioning it explicitly.

Lemma 3.4. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. Then for every length $L \in \mathbb{N}_0$ and every point $p \in \Gamma(G, X \cup \mathcal{H})$ there are only finitely many cycles γ of length at most L in $\Gamma(G, X \cup \mathcal{H})$ such that γ starts at p and the components of γ are isolated.*

Proof. Suppose that there are infinitely many cycles starting at some point p whose components are isolated. Since X and Λ are finite, there has to be an index $\lambda \in \Lambda$

such that $\widetilde{H}_\lambda$ contains infinitely many different labels of edges of these cycles. Then their length is growing unboundedly with respect to X. But this cannot happen due to Lemma 2.12. □

Given a number $n \in \mathbb{N}$, our goal is it to construct an infinite sequence of words of length at least n which satisfies the ag-condition. First we introduce a method that allows us to combine to sequences of words which satisfy the ag-condition to a sequence of longer words that satisfies the ag-condition.

Lemma 3.5. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. Let $M, N \geqslant 2$ be two numbers and let $(v_n)_{n\in\mathbb{N}}$ respectively $(w_n)_{n\in\mathbb{N}}$ be sequences of words of length M respectively N over the alphabet $X \cup \mathcal{H}$. We write $v_n = v_1^{(n)} \cdots v_M^{(n)}$ and $w_n = w_1^{(n)} \cdots w_N^{(n)}$. If $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ satisfy the ag-condition then the following hold.*

1) *Suppose that there are two strictly increasing sequences of natural numbers $(s_i)_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$ such that $v_M^{(s_i)}$ and $w_1^{(t_i)}$ never lie in the same set X, nor represent elements of the same group H_λ. Then there is a strictly increasing sequence of natural numbers $(k_i)_{i\in\mathbb{N}}$ such that the sequence of words*
$$(v_1^{(s_{k_i})} \cdots v_M^{(s_{k_i})} w_1^{(t_{k_i})} \cdots w_N^{(t_{k_i})})_{i\in\mathbb{N}}$$
satisfies the ag-condition.
2) *Suppose that there are two strictly increasing sequences of natural numbers $(s_i)_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$ such that $v_M^{(s_i)}$ and $w_1^{(t_i)}$ lie in the same group $\widetilde{H}_\lambda$. If the sequence of lengths $(|v_M^{(s_i)} w_1^{(t_i)}|_X)_{i\in\mathbb{N}}$ is unbounded, then there is a strictly increasing sequence of natural numbers $(k_i)_{i\in\mathbb{N}}$ such that the sequence of words*
$$(v_1^{(s_{k_i})} \cdots v_{M-1}^{(s_{k_i})} z^{(i)} w_2^{(t_{k_i})} \cdots w_N^{(t_{k_i})})_{i\in\mathbb{N}},$$
where $z^{(i)} \in \widetilde{H}_\lambda$ is the element represented by the word $v_M^{(s_{k_i})} w_1^{(t_{k_i})}$, satisfies the ag-condition.

Proof. First we prove 1). Suppose that there is no sequence such $(k_i)_{i\in\mathbb{N}}$. Then there are infinitely many words of the form
$$(v_1^{(s_i)} \cdots v_M^{(s_i)} w_1^{(t_i)} \cdots w_N^{(t_i)})_{i\in\mathbb{N}}$$
which do not satisfy the conditions of Definition 3.2. Due to Remark 3.3 we can assume that all but the second condition are satisfied. Thus by restriction to a subsequence we can assume that none of the words
$$v_1^{(s_i)} \cdots v_M^{(s_i)} w_1^{(t_i)} \cdots w_N^{(t_i)}$$
has regular neighbourhood. Since the subwords $v_1^{(s_i)} \cdots v_M^{(s_i)}$ and $w_1^{(t_i)} \cdots w_N^{(t_i)}$ have regular neighbourhoods we obtain cycles
$$q_i = v_{a_i}^{(s_i)} \cdots v_M^{(s_i)} w_1^{(t_i)} \cdots w_{b_i}^{(t_i)} u_i \tag{3.1}$$
for some appropriate elements $u_i \in X \cup \mathcal{H}$ and $a_i, b_i \in \mathbb{N}$. Let a_i respectively b_i be maximal respectively minimal for each $i \in \mathbb{N}$ with the property that there is a cycle of the form (3.1). Then the components of q_i are isolated and consist of single edges. Due to our construction the length of the cycles q_i is bounded by

$M+N+1$. In this case Lemma 3.4 tells us that there are only finitely many such cycles. This contradicts the fact that every cycle q_i contains an edge with the label $w_1^{(t_i)}$. Indeed, the conditions 4) and 5) of Definition 3.2 immediately imply that the sequence $(|w_1^{(t_i)}|_X)_{i\in\mathbb{N}}$ is unbounded.
Case 2) will now be reduced to case 1). Suppose that there is no such sequence $(k_i)_{i\in\mathbb{N}}$. Then there are infinitely many words of the form

$$(v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)}w_2^{(t_i)}\cdots w_N^{(t_i)})_{i\in\mathbb{N}}$$

with $z^{(i)}=v_M^{(s_i)}w_1^{(t_i)}$ which do not satisfy the conditions of Definition 3.2. Again, by applying Remark 3.3 we can assume that all but the second condition are satisfied. Suppose that infinitely many subwords $v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)}$ do not have a regular neighbourhood. Then, by restriction to a subsequence, we can assume that none of the words $v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)}$ has a regular neighbourhood. Since the word $v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}$ has a regular neighbourhood for every $i\in\mathbb{N}$, we can choose $a_i\in\mathbb{N}$ maximal such that there is a cycle of the form $q_i=v_{a_i}^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)}u_i$ for some appropriate element $u_i\in X\cup\mathcal{H}$. Due to the maximality of a_i the components of q_i are isolated and consist of single edges. Further we see that the length of every cycle q_i is bounded by $M+1$. Thus, as in case 1), Lemma 3.4 implies that there are only finitely many cycles q_i. But this contradicts the assumption that $z^{(i)}$ is contained in q_i, since the sequence $(|z^{(i)}|_X)_{i\in\mathbb{N}}$ is unbounded.

In particular we see that, by restriction to a subsequence, the words

$$v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)} \text{ and } w_2^{(t_i)}\cdots w_N^{(t_i)}$$

have regular neighbourhoods and the sequences

$$(v_1^{(s_i)}\cdots v_{M-1}^{(s_i)}z^{(i)})_{i\in\mathbb{N}} \text{ and } (w_2^{(t_i)}\cdots w_N^{(t_i)})_{i\in\mathbb{N}}$$

satisfy the ag-condition. Due to construction $z^{(i)}$ and $w_2^{(t_i)}$ neither lie both in the relative generating set X nor they lie in the same group $\widetilde{H}_\lambda$. Thus the conditions of case 1) are satisfied and the claim follows. ☐

Corollary 3.6. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda=\{H_\lambda\mid\lambda\in\Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of words of length $m\geqslant 2$ over the alphabet $X\cup\mathcal{H}$ which satisfies the ag-condition and let K be the subgroup of G generated by $\{\overline{w_n}\mid n\in\mathbb{N}\}$. Then there is a constant $C\in\mathbb{N}$ such that for every natural number $L\in\mathbb{N}$ there is a sequence of words $(v_n)_{n\in\mathbb{N}}$ over $X\cup\mathcal{H}$ such that the following holds.*

1) *$(v_n)_{n\in\mathbb{N}}$ satisfies the ag-condition.*
2) *The length of every word v_n is bounded by $L\leqslant\|v_n\|\leqslant L+C$.*
3) *Every word v_n represents an element of the group K.*

Proof. The definition of the ag-condition implies, that by restriction to a subsequence, we can assume that $(w_n)_{n\in\mathbb{N}}$ has one of the following two properties.

1) There is no $n\in\mathbb{N}$ such that $w_1^{(n)}$ and $w_m^{(n)}$ represent elements of the same parabolic subgroup H_λ.

2) For every $n \in \mathbb{N}$ the elements $w_1^{(n)}$ and $w_m^{(n)}$ represent elements of the same parabolic subgroup H_λ and satisfy $|w_1^{(n)}|_X, |w_m^{(n)}|_X \geqslant n$.

We will use Lemma 3.5 inductively. We assume that the first case is satisfied. Then due to Lemma 3.5 1) there are subsequences

$$(w_1^{(i_n)} \cdots w_m^{(i_n)})_{n\in\mathbb{N}} \text{ and } (w_1^{(j_n)} \cdots w_m^{(j_n)})_{n\in\mathbb{N}}$$

such that the concatenated sequence

$$(v_n)_{n\in\mathbb{N}} := (w_1^{(i_n)} \cdots w_m^{(i_n)} w_1^{(j_n)} \cdots w_m^{(j_n)})_{n\in\mathbb{N}}$$

satisfies the ag-condition. Further we have $\| v_n \| = 2m$ and Lemma 3.5 1) can be applied to the sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_1^{(j_n)} \cdots w_m^{(j_n)})_{n\in\mathbb{N}}$. Given a number $k \in \mathbb{N}$, we can construct inductively a sequence of length km that satisfies the ag-condition.

In the case of 2) the unbounded growth of $|w_1^{(n)}|_X$ implies that there is a subsequence $(w_1^{(k_n)} \cdots w_m^{(k_n)})_{n\in\mathbb{N}}$ such that $|w_m^{(n)} w_1^{(k_n)}|_X \geqslant n$ for every $n \in \mathbb{N}$. In particular we see that Lemma 3.5 2) can be applied to the sequences $(w_n)_{n\in\mathbb{N}}$ and $(w_{k_n})_{n\in\mathbb{N}}$ to obtain a sequence of the form

$$(v_n)_{n\in\mathbb{N}} := (w_1^{(i_n)} \cdots w_{m-1}^{(i_n)} z^{(n)} w_2^{(j_n)} \cdots w_m^{(j_n)})_{n\in\mathbb{N}}$$

which satisfies the ag-condition. Here the letter $z^{(n)} \in \widetilde{H}_\lambda$ corresponds to some element $\overline{w_m^{(i_n)} w_1^{(j_n)}} \in H_\lambda$. Now we can apply Lemma 3.5 2) to $(v_n)_{n\in\mathbb{N}}$ and an appropriate subsequence $(w_{r_n})_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$. By induction we obtain sequences of words of length $m + k(m-1)$ which satisfy the ag-condition. The claim now follows by setting $C = m$. □

Lemma 3.7. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. If $(w_n)_{n\in\mathbb{N}}$ is a sequence of words over $X \cup \mathcal{H}$ which satisfies the ag-condition, then the subgroup K of G generated by $\{\overline{w_n} \mid n \in \mathbb{N}\}$ is unbounded with respect to $d_{X\cup\mathcal{H}}$.*

Proof. Suppose that this is not the case. Then there is a number $N \in \mathbb{N}$ such that the $d_{X\cup\mathcal{H}}$-length of every element of K is bounded by N. Due to Corollary 3.6 there is a number $L \geqslant 4N$ such that there is a sequence of words $(v_n)_{n\in\mathbb{N}}$ of length L over $X \cup \mathcal{H}$ such that $(v_n)_{n\in\mathbb{N}}$ satisfies the ag-condition and every word v_n represents an element of K. By restriction to a subsequence we can assume that $|v_n|_{X\cup\mathcal{H}} = M \leqslant N$ for every $n \in \mathbb{N}$ and some appropriate $M \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$ there is a geodesic $u_1^{(n)} \cdots u_M^{(n)}$ such that $q_n = v_1^{(n)} \cdots v_L^{(n)} u_1^{(n)} \cdots u_M^{(n)}$ is a cycle in $\Gamma(G, X \cup \mathcal{H})$. The ag-condition assures that $v_1^{(n)} \cdots v_L^{(n)}$ has a regular neighbourhood. In particular the components of $v_1^{(n)} \cdots v_L^{(n)}$ are isolated and consist of single edges. Further the ag-condition tells us that at least every second edge is isolated. Thus there are at least $2N$ isolated components in $v_1^{(n)} \cdots v_L^{(n)}$. We note that for every isolated component $u_i^{(n)}$ in $u_1^{(n)} \cdots u_M^{(n)}$ there is at most one isolated component $v_j^{(n)}$ in $v_1^{(n)} \cdots v_L^{(n)}$ which is connected to $u_i^{(n)}$, since otherwise there would be a connection between two different isolated components of $v_1^{(n)} \cdots v_L^{(n)}$ by a H_λ-path in $\Gamma(G, X \cup \mathcal{H})$. This implies that there are at least $2N - M \geqslant N$ isolated components in q_n. Due to the ag-condition these components grow unboundedly. But since the $d_{X\cup\mathcal{H}}$-length of every cycle q_n is bounded by $N + L$, we obtain a contradiction to Lemma 3.4. $\square$

Lemma 3.8. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. Let $K \leqslant G$ be an infinite subgroup which is bounded with respect to the metric $d_{X\cup\mathcal{H}}$. Then there is an element $g \in G$ and an index $\eta \in \Lambda$ such that $|gKg^{-1} \cap H_\eta| = \infty$.*

Proof. Since K is bounded with respect to $d_{X\cup\mathcal{H}}$ every conjugate gKg^{-1} is a bounded subset of $\Gamma(G, X \cup \mathcal{H})$. Let $n \in \mathbb{N}$ be minimal with the following property:

($\star$) There is a conjugate $H = gKg^{-1}$ of K and a finite relative generating set X of G such that there is an infinite sequence of pairwise distinct elements $k_j \in H$ of the form
$$k_j = u_1^{(j)} \cdots u_n^{(j)}$$
such that the words $u_1^{(j)} \cdots u_n^{(j)}$ are labels of geodesics in $\Gamma(G, X \cup \mathcal{H})$.

Since G is finitely generated we can enlarge the finite relative generating set of G, which was chosen with respect to n, to a finite ordinary generating set $X = X^{-1}$ of G. Indeed, this is possible due to Theorem 2.13 since the minimality of n allows us to take a subsequence $(k_j)_{j\in\mathbb{N}}$ which also satisfies the condition on n. First we consider the case that $n = 1$. In this case the finiteness of X implies that there are infinitely many elements k_j lying in $\mathcal{H} = \bigcup_{\lambda\in\Lambda} (\widetilde{H}_\lambda \backslash \{1\})$. Further the index set Λ has to be finite since G is finitely generated ([Osi06, Corollary 2.48]). Thus there has to be an index $\eta \in \Lambda$ such that infinitely many pairwise distinct elements of the sequence $(k_j)_{j\in\mathbb{N}}$ lie in $\widetilde{H}_\eta$. In this case, the intersection of $H = gKg^{-1}$ and the parabolic subgroup H_η is infinite. Next we will show that the assumption $n \geqslant 2$ leads to a contradiction. Thus, let $n \geqslant 2$. Our goal is to modify the sequence $(k_j)_{j\in\mathbb{N}}$ in such a way that the ag-condition is satisfied. Suppose that there are infinitely many $j \in \mathbb{N}$ such that $u_1^{(j)} \in X$. Since X is finite there is an element $x_1 \in X$ and a subsequence $(k_{j_m})_{m\in\mathbb{N}}$ of $(k_j)_{j\in\mathbb{N}}$ with $u_1^{(j_m)} = x_1$ for every m. Otherwise, if there are only finitely many j with $u_1^{(j)} \in X$, the finiteness of Λ implies that there is an index $\lambda_1 \in \Lambda$ and a subsequence $(k_{j_m})_{m\in\mathbb{N}}$ with $u_1^{(j_m)} \in H_{\lambda_1}$ for every m. In this case there could be an element $h_1 \in H_{\lambda_1}$ such that $u_1^{(j_m)} = h_1$ for infinitely many m. Then we can add the elements h_1 and h_1^{-1} to X and proceed as in the first case by restricting $(k_j)_{j\in\mathbb{N}}$ to a subsequence $(k_{l_m})_{m\in\mathbb{N}}$ such that on the one hand the condition ($\star$) is satisfied and on the other hand $u_1^{(l_m)} = h_1$ for every $m \in \mathbb{N}$. In the remaining case we can choose a subsequence $(k_{j_m})_{m\in\mathbb{N}}$ such that $|u_i^{(j_m)}|_X < |u_i^{(j_{m+1})}|_X$ for every $m \in \mathbb{N}$. In every case we replace the given sequence with the subsequence. We proceed analogously with the other indices $i \in \{2, \ldots, n\}$ and we write $(g_j)_{j\in\mathbb{N}} = (u_1^{(j)} \cdots u_n^{(j)})_{j\in\mathbb{N}}$ for the sequence of elements $g_j \in H$ obtained in that way. Further we can assume that neither the two consecutive letters $u_i^{(j)}, u_{i+1}^{(j)}$ nor the letters $u_n^{(j)}, u_1^{(j)}$ both lie in X, since otherwise we could add $u_i^{(j)} u_{i+1}^{(j)}$ and $(u_i^{(j)} u_{i+1}^{(j)})^{-1}$ (respectively $u_n^{(j)} u_1^{(j)}$

and $(u_n^{(j)}u_1^{(j)})^{-1})$ to X in order to obtain a shorter sequence of infinitely many pairwise distinct elements of H (respectively of $u_{-1}^{(j)}Hu_1^{(j)}$) with respect to $d_{X\cup\mathcal{H}}$. Since the words $u_1^{(j)}\cdots u_n^{(j)}$ are labels of geodesics, it follows immediately that two consecutive letters $u_i^{(j)}$ and $u_{i+1}^{(j)}$ do not represent elements of the same parabolic subgroup H_λ. To get sure that the sequence $(g_j)_{j\in\mathbb{N}}$ satisfies the forth condition of the definition of the ag-condition, an element $u_1^{(j)}\cdots u_n^{(j)}$ can be replaced by its inverse $(u_1^{(j)}\cdots u_n^{(j)})^{-1} = (u_n^{(j)})^{-1}\cdots(u_1^{(j)})^{-1}$. Due to Remark 3.3 a further restriction to an appropriate subsequence provides us with a sequence $(g_j)_{j\in\mathbb{N}}$ which satisfies the ag-condition. But this contradicts Lemma 3.7 since H was supposed to be bounded with respect to $d_{X\cup\mathcal{H}}$. □

Theorem 3.9. *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of parabolic subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{rel}_{G,H_\Lambda}$ is well defined. Let $K \leqslant G$ be a subgroup which is bounded with respect to the relative metric $d_{X\cup\mathcal{H}}$. Then K is either finite or conjugated to an infinite subgroup of a parabolic group.*

Proof. Suppose that K is an infinite group. Due to Theorem 3.8 there is an index $\eta \in \Lambda$ and an element $g \in G$ with $|gKg^{-1} \cap H_\eta| = \infty$. Thus we can choose a sequence of distinct, nontrivial elements $(h_n)_{n\in\mathbb{N}}$ of $gKg^{-1} \cap H_\eta$. Suppose that gKg^{-1} is not a subgroup of H_η. Then there is an element $a \in gKg^{-1}\backslash H_\eta$. By adding a, a^{-1} to X we can ensure that the $d_{X\cup\mathcal{H}}$-length of every element of the sequence $(h_na)_{n\in\mathbb{N}}$ is bounded by 2. It is easily seen that, by restriction to an appropriate subsequence, the sequence of word over $X \cup \mathcal{H}$ corresponding to $(h_na)_{n\in\mathbb{N}}$ satisfies the ag-condition. In this case Lemma 3.7 says that the subgroup $\langle\{ah_n \mid n \in \mathbb{N}\}\rangle$ of gKg^{-1} is unbounded with respect to $d_{X\cup\mathcal{H}}$. This is a contradiction to the assumption that K is bounded and thus the claim follows. □

References

[GM18] Daniel Groves and Jason Fox Manning. Dehn fillings and elementary splittings. *Trans. Amer. Math. Soc.*, 370(5):3017–3051, 2018.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.